Let the Transformations Begin!

5.3

Translations of Linear and Exponential Functions

LEARNING GOALS

In this lesson, you will:

- Translate linear and exponential functions vertically.
- Translate linear and exponential functions horizontally.

KEY TERMS

- basic function
- transformation
- vertical translation
- coordinate notation
- argument of a function
- horizontal translation

Computer programmers write and use functions in much the same way as you do in mathematics. For example, a programmer may write the function **fabs(x)** in a programming language. This function calculates the absolute value of the variable *x*.

The name of the function is "fabs," and the variable inside the parentheses is known as the argument. Input is "passed" (either from the program itself or from a user of the program) to the argument, and the function "returns" an output—in this case, the absolute value of *x*.

How is this programming function related to the functions you have been studying?

PROBLEM 1 Vertical Translations

Consider the three linear functions shown.

- $g(x) = x$
- $c(x) = (x) + 3$
- $d(x) = (x) - 3$

The first function is the *basic function*. A **basic function** is the simplest function of its type. In this case, $g(x) = x$ is the simplest linear function. It is in the form $f(x) = ax + b$, where $a = 1$ and $b = 0$.

You can write the given functions $c(x)$ and $d(x)$ in terms of the basic function $g(x)$. For example, because $g(x) = x$, you can substitute $g(x)$ for x in the equation for $c(x)$, as shown.

$$c(x) = (x) + 3$$
$$\downarrow$$
$$c(x) = g(x) + 3$$

1. Write the function $d(x)$ in terms of the basic function $g(x)$.

   $d(x) =$ ________________

2. Describe the operation performed on the basic function $g(x)$ to result in each of the equations for $c(x)$ and $d(x)$.

3. Use a graphing calculator to graph each function with the bounds $[-10, 10] \times [-10, 10]$. Then, sketch the graph of each function. Label each graph.

4. Compare the y-intercepts of the graphs of $c(x)$ and $d(x)$ to the y-intercept of the basic function $g(x)$. What do you notice?

5. Write the y-value of each ordered pair for the three given functions.

$g(x) = x$	$c(x) = (x) + 3$	$d(x) = (x) - 3$
(−2, ____)	(−2, ____)	(−2, ____)
(−1, ____)	(−1, ____)	(−1, ____)
(0, ____)	(0, ____)	(0, ____)
(1, ____)	(1, ____)	(1, ____)
(2, ____)	(2, ____)	(2, ____)

6. Use the table to compare the ordered pairs of the graphs of $c(x)$ and $d(x)$ to the ordered pairs of the graph of the basic function $g(x)$. What do you notice?

A **transformation** is the mapping, or movement, of all the points of a figure in a plane according to a common operation. A **vertical translation** is a type of transformation that shifts the entire graph up or down. A vertical translation affects the y-coordinate of each point on the graph.

Coordinate notation is a notation that uses ordered pairs to describe a transformation in a coordinate plane. For example, you can use the coordinate notation shown to indicate a vertical translation.

$(x, y) \rightarrow (x, y + b)$, where b is a real number.

7. Use coordinate notation to represent the vertical translation of each function.

- $g(x) = x$
  (x, y)
- $c(x) = (x) + 3$
  $(x, y) \rightarrow$ ____________
- $d(x) = (x) - 3$
  $(x, y) \rightarrow$ ____________

Now, let's consider the three exponential functions shown.

- $h(x) = 2^x$
- $s(x) = (2^x) + 3$
- $t(x) = (2^x) - 3$

In this case, $h(x) = 2^x$ is the basic function because it is the simplest exponential function with a base of 2. It is in the form $f(x) = a \cdot b^x$, where $a = 1$ and $b = 2$.

8. Write the functions $s(x)$ and $t(x)$ in terms of the basic function $h(x)$. Then, describe the operation performed on the basic function $h(x)$ to result in each of the equations for $s(x)$ and $t(x)$.

$s(x) =$ ____________________

$t(x) =$ ____________________

9. Use a graphing calculator to graph each function with the bounds $[-10, 10] \times [-10, 10]$. Then, sketch the graph of each function. Label each graph.

5

10. Compare the y-intercepts of the graphs of $s(x)$ and $t(x)$ to the y-intercept of the graph of the basic function $h(x)$. What do you notice? Are the results the same as when you compared the graphs of the linear functions in Question 4?

11. Write the y-value of each ordered pair for the three given functions.

$h(x) = 2^x$	$s(x) = (2^x) + 3$	$t(x) = (2^x) - 3$
(−2, ___)	(−2, ___)	(−2, ___)
(−1, ___)	(−1, ___)	(−1, ___)
(0, ___)	(0, ___)	(0, ___)
(1, ___)	(1, ___)	(1, ___)
(2, ___)	(2, ___)	(2, ___)

12. Use the table to compare the ordered pairs of the graphs of $s(x)$ and $t(x)$ to the ordered pairs of the graph of the basic function $h(x)$. What do you notice? Are the results the same as when you compared the y-values for the linear functions in Question 6?

13. Explain how you know that the graphs of $s(x)$ and $t(x)$ are vertical translations of the graph of $h(x)$.

5

14. Use coordinate notation to represent the vertical translation of each function.

- $h(x) = 2^x$
  (x, y)
- $s(x) = (2^x) + 3$
  $(x, y) \rightarrow$ ________________
- $t(x) = (2^x) - 3$
  $(x, y) \rightarrow$ ________________

15. Describe each graph in relation to its basic function.

a. Compare $f(x) = x + b$ to the basic function $g(x) = x$ for $b > 0$.

b. Compare $f(x) = x + b$ to the basic function $g(x) = x$ for $b < 0$.

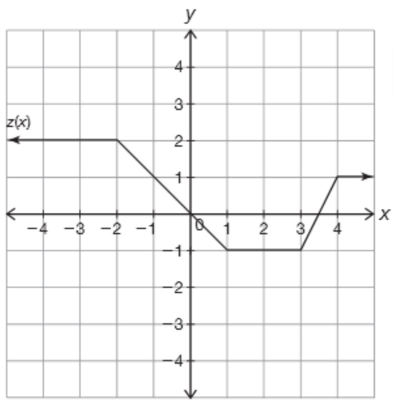
c. Compare $f(x) = b^x + k$ to the basic function $h(x) = b^x$ for $k > 0$.

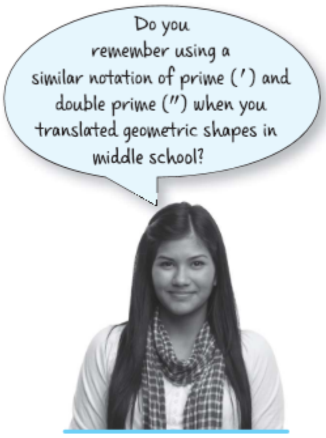
d. Compare $f(x) = b^x + k$ to the basic function $h(x) = b^x$ for $k < 0$.

16. The graph of a function $z(x)$ is shown. Sketch the graphs of $z'(x)$ and $z''(x)$.

a. $z'(x) = z(x) + 3$

b. $z''(x) = z(x) - 4$





5

17. Write the equation of each function after a vertical translation down 6 units.

a. $p(x) = 5x$

$p'(x) =$ ________________

b. $q(x) = 3x^2$

$q'(x) =$ ________________

c. $r(x) = \frac{1}{2}x^3$

$r'(x) =$ ________________

PROBLEM 2 Horizontal Translations

Consider the three exponential functions shown, where $h(x) = 2^x$ is the basic function.

- $h(x) = 2^x$
- $v(x) = 2^{(x+3)}$
- $w(x) = 2^{(x-3)}$

In Problem 1 *Vertical Translations*, the operations that produced the vertical translations were performed on the function $h(x)$. That is, 3 was added to $h(x)$ and 3 was subtracted from $h(x)$. In this problem, the operations are performed on x, which is the *argument* of the function. The **argument of a function** is the *variable* on which the function operates. So, in this case, 3 is added to x and 3 is subtracted from x.

You can write the given functions $v(x)$ and $w(x)$ in terms of the basic function $h(x)$. To write $v(x)$ in terms of $h(x)$, you just substitute $x + 3$ into the argument for $h(x)$, as shown.

$$h(x) = 2^x$$

$$v(x) = h(x + 3) = 2^{(x+3)}$$

So, $x + 3$ replaces the variable x in the function $h(x) = 2^x$.

1. Write the function $w(x)$ in terms of the basic function $h(x)$.

2. Use a graphing calculator to graph each function with the bounds $[-10, 10] \times [-10, 10]$. Then, sketch the graph of each function. Label each graph.

5

3. Compare the graphs of $v(x)$ and $w(x)$ to the graph of the basic function. What do you notice?

4. Write the x-value of each ordered pair for the three given functions. You can use your graphing calculator to determine the x-values.

$h(x) = 2^x$	$v(x) = 2^{(x+3)}$	$w(x) = 2^{(x-3)}$
(____, $\frac{1}{4}$)	(____, $\frac{1}{4}$)	(____, $\frac{1}{4}$)
(____, $\frac{1}{2}$)	(____, $\frac{1}{2}$)	(____, $\frac{1}{2}$)
(____, 1)	(____, 1)	(____, 1)
(____, 2)	(____, 2)	(____, 2)
(____, 4)	(____, 4)	(____, 4)

5. Use the table to compare the ordered pairs of the graphs of $v(x)$ and $w(x)$ to the ordered pairs of the graph of the basic function $h(x)$. What do you notice?

A **horizontal translation** of a graph is a shift of the entire graph left or right. A horizontal translation affects the x-coordinate of each point on the graph.

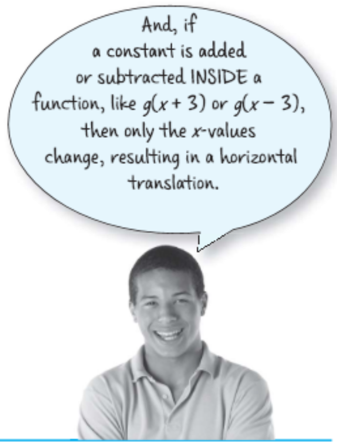
You can use the coordinate notation shown to indicate a horizontal translation.

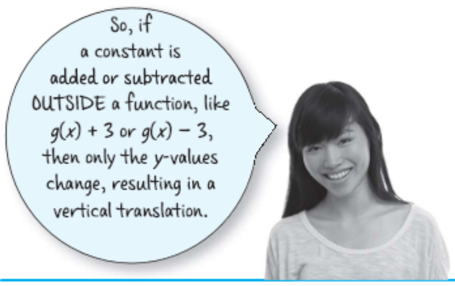
$$(x, y) \rightarrow (x + a, y), \text{ where } a \text{ is a real number.}$$

5

6. Use coordinate notation to represent the horizontal translation of each function.

- $h(x) = 2^x$
  (x, y)
- $v(x) = 2^{(x+3)}$
  $(x, y) \rightarrow$ __________
- $w(x) = 2^{(x-3)}$
  $(x, y) \rightarrow$ __________





7. Describe each graph in relation to its basic function.

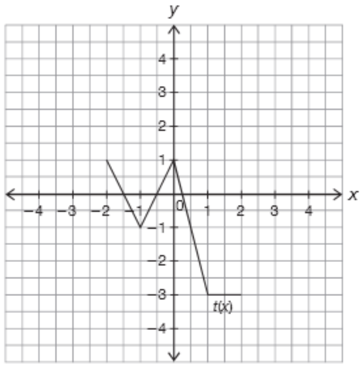
a. Compare $f(x) = b^{x-c}$ to the basic function $h(x) = b^x$ for $c > 0$.

b. Compare $f(x) = b^{x-c}$ to the basic function $h(x) = b^x$ for $c < 0$.

8. The graph of a function $t(x)$ is shown. Sketch the graphs of $t'(x)$ and $t''(x)$.

a. $t'(x) = t(x + 3)$

b. $t''(x) = t(x - 1)$



9. Write the equation of each function after a horizontal translation left 10 units.

a. $p(x) = 5x$

$p'(x) =$ ________________

b. $q(x) = 3x^2$

$q'(x) =$ ________________

c. $r(x) = \frac{1}{2}x^3$

$r'(x) =$ ________________

5

PROBLEM 3 The Curious Case of Linear Functions

Consider the three linear functions shown, where $j(x)$ and $k(x)$ are translations of $g(x)$.

- $g(x) = x$
- $j(x) = g(x) + 5$
- $k(x) = g(x + 5)$

1. Describe the translation of the graph of $g(x)$ that produces $j(x)$. Then, describe the translation of the graph of $g(x)$ that produces $k(x)$.

2. Which function has an operation performed on $g(x)$? Which function has an operation performed on the argument of $g(x)$?

5

3. Rewrite $j(x)$ so that it is in terms of x instead of in terms of $g(x)$. Then, rewrite $k(x)$ so that it is in terms of x. What do you notice?

Kyle, Turk, and Tobias are discussing about whether the translations that produce $j(x)$ and $k(x)$ in Question 1 are the same.

Kyle

The translation that produces $j(x)$ is the same as the translation that produces $k(x)$ because, algebraically, both functions simplify to $x + 5$.

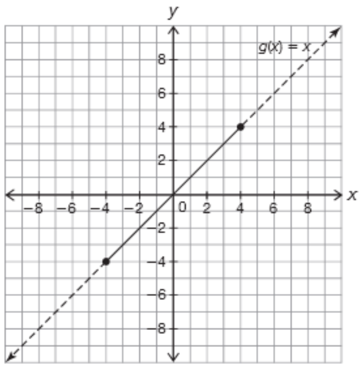
Turk

The translation that produces $j(x)$ is the same as the translation that produces $k(x)$ because when I graph them on my calculator, I get the same line.

Tobias

The translation that produces $j(x)$ is not the same as the translation that produces $k(x)$ because $j(x)$ is a vertical translation and $k(x)$ is a horizontal translation.

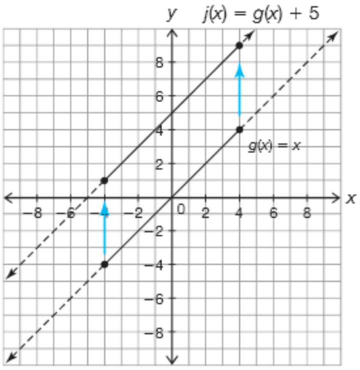
To see why Tobias is correct, consider a segment of the basic function $g(x) = x$. The function $g(x)$ is shown as a dotted line, and the segment of the function is shown as a solid segment.

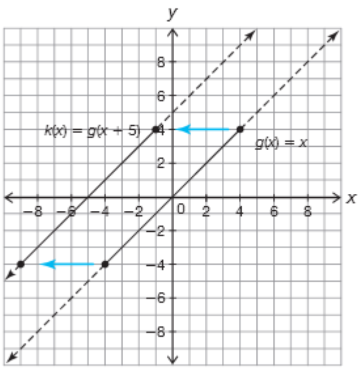


5

The function $j(x) = g(x) + 5$ is a vertical translation that is 5 units up from the graph of $g(x)$.

The function $k(x) = g(x + 5)$ is a horizontal translation that is 5 units to the left of the graph of $g(x)$.





Notice that $j(x)$ and $k(x)$ are both part of the same line. However, the segments are in different locations on the coordinate plane.

So, Tobias' reasoning is correct because the translation that produces $j(x)$ is not the same as the translation that produces $k(x)$, even though the algebraic representation and the graphs of the lines are the same.

Talk the Talk

1. Match each function form with its corresponding graph.

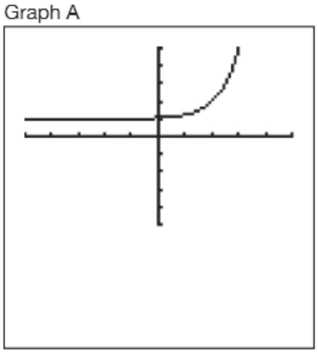
- $f(x) = ax + b$, where $a = 1$ and $b > 0$

- $f(x) = ax + b$, where $a = 1$ and $b < 0$

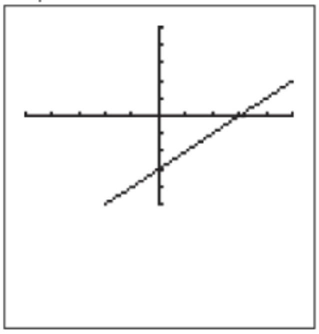
- $f(x) = b^{x-c} + k$, where $c > 0$ and $k > 0$

- $f(x) = b^{x-c} + k$, where $c < 0$ and $k < 0$

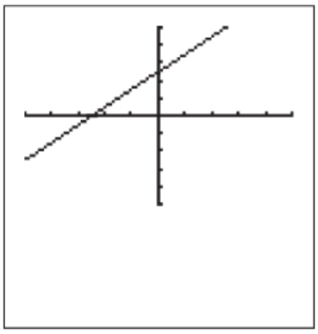
Graph A



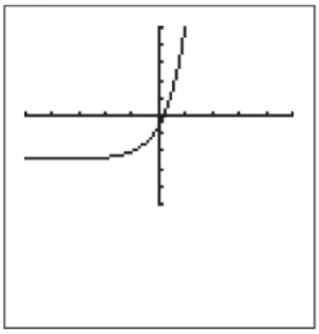
Graph B



Graph C



Graph D



5

2. Complete the table by describing the graph of each function as a transformation of its basic function.

Function Form	Equation Information	Description of Transformation of Graph
$f(x) = (x) + b$	$b > 0$	
	$b < 0$	
$f(x) = (x - b)$	$b > 0$	
	$b < 0$	
$f(x) = b^x + k$	$b > 1, k > 0$	
	$b > 1, k < 0$	
$f(x) = b^{x-c}$	$b > 1, c > 0$	
	$b > 1, c < 0$	

5

Be prepared to share your solutions and methods.